Assignment # 4	Due on November 27th in class

1. On the vector space X, suppose there are two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on it. If these two norms are equivalent, prove that $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

2. On finite dimensional vector space $\mathbb{R}^n$, define the norm as

$$\|x\| = |x_1| + \cdots + |x_n| \ .$$

a) Prove that the above defined norm is well-defined. In other words, it satisfies the definition of norm (the properties of norm).

b) Prove that under the above defined norm, $(\mathbb{R}^n, \|\cdot\|)$ is complete. (You can use the following fact without needing to prove it: $\mathbb{R}$ is complete)

Solution:

1.

Proof. As the two norms are equivalent, there exists $0 < \alpha < \beta$, such that

$$\alpha \|x\|_2 \leq \|x\|_1 \leq \beta \|x\|_2 .$$

It then follows that a sequence $x_1, x_2, \cdots$ is a Cauchy sequence in $(X, \|\|_1)$ if and only if it is a Cauchy sequence in $(X, \|\|_1)$, which finishes the proof. □

2.

a)

The proof is almost routine, we will just verify the triangle inequality here.

For $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}\|x+y\| &= |x_1+y_1| + \cdots + |x_n+y_n| \\ &\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| \\ &= (|x_1| + \cdots + |x_n|) + (|y_1| + \cdots + |y_n|) \\ &= \|x\| + \|y\| .\end{aligned}$$

b) Sketchy proof: Assume that we have a Cauchy sequence in $\mathbb{R}^n$, then the 1st coordinates still form a Cauchy sequence in $\mathbb{R}$. As $\mathbb{R}$ is complete, it must converge to some y_1. Similary, the k-th coordinates must converge to y_k for all $1 \leq k \leq n$. It then remains to show that $(y_1, \cdots, y_n)$ is the desired limit of the original Cauchy sequence in $\mathbb{R}^n$.